M.Sc III sem.

Paper – XI (MAT CC – 10)

Normed linear space:— Let N be a complex (or real) linear space and let $\| \ \|$ be a function from N into R such that for all $x, y \in N$ and all $\alpha \in C$ (or R), we have

(n_1) : $\|x\| \geq 0$

(n_2) : $\|x\| = 0 \Leftrightarrow x = 0$

(n_3) : $\|x+y\| \leq \|x\| + \|y\|$

(n_4) : $\|\alpha x\| = |\alpha| \|x\|$.

Then the function $\| \ \|$ is called a norm N and the pair $(N, \| \ \|)$ is called a complex (or real) normed linear space.

Theorem:— Let N be a normed linear space and let d be a function from $N \times N$ into R defined by $d(x, y) = \|x-y\|$. Then d is a metric on N.

Proof:— We shall verify the axioms of above definition.

(m_1) : $x, y \in N \Rightarrow x-y \in N \Rightarrow \|x-y\| \geq 0$ by (n_1)

$\Rightarrow d(x, y) \geq 0$

(m_2) : $d(x, y) = 0 \Rightarrow \|x-y\| = 0 \Rightarrow x-y = 0$ by (n_2)

$\Rightarrow x = y$.

(m_3) : $d(x, y) = \|x-y\| = \|(-1)(y-x)\| = |(-1)| \|y-x\|$ by (n_4) $= \|y-x\| = d(y, x)$

(m_4) : If $x, y, z \in N$, then

$\|x-y\| = \|x-z+z-y\| \leq \|x-z\| + \|z-y\|$ (by n_3)

Hence $d(x, y) \leq d(x, z) + d(z, y)$

it follows that d is a metric on N.

Proved.

Navneet

Teacher's Signature

Scanned By Camera Scanner

Banach Space or B-Space:

A normed linear space which is complete as a metric space is called a Banach space or B-space.

Example of Banach Spaces

1. Show that the real linear space R and the complex linear space C are Banach spaces under the norm

$$\|x\| = |x|;\ x \in C\ (\text{or } R)$$

Solution:— Since $\|x\| = |x| \geq 0$ and $\|x\| = 0 \Rightarrow |x| = 0 \Rightarrow x = 0$ (Axioms (N1) and (N2))

(N3): Let $z, w \in C$ and let $\bar{z}$ and $\bar{w}$ denote their complex conjugates then

$$|z+w|^2 = (z+w)(\bar{z}+\bar{w})$$

$$= z\bar{z} + z\bar{w} + \bar{z}w + w\bar{w} \leq |z|^2 + 2|z\bar{w}| + |w|^2 = |z|^2 + 2|z|\,|w| + |w|^2$$

$$(\because |z\bar{w}| = |zw| = |z|\,|w|)$$

$$= (|z| + |w|)^2$$

Hence $|z+w| \leq |z| + |w|$ that is $\|z+w\| \leq \|z\| + \|w\|$.

(N4): $\|\alpha x\| = |\alpha x| = |\alpha|\,|x| = |\alpha|\,\|x\|$.

Thus C and R are both normed linear spaces. Also we know that these space are complete. Hence they are Banach spaces.

Ashwini Kumar Singh.